Characters and composition factor multiplicities for the Lie superalgebra $\mathfrak{gl}(m/n)$

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Abstract

The multiplicities $a_{\lambda,\mu}$ of simple modules L_μ in the composition series of Kac modules V_λ for the Lie superalgebra $\mathfrak{gl}(m/n)$ were described by Serganova, leading to her solution of the character problem for $\mathfrak{gl}(m/n)$. In Serganova's algorithm all μ with nonzero $a_{\lambda,\mu}$ are determined for a given λ; this algorithm turns out to be rather complicated. In this Letter a simple rule is conjectured to find all nonzero $a_{\lambda,\mu}$ for any given weight μ. In particular, we claim that for an r-fold atypical weight μ there are 2^r distinct weights λ such that $a_{\lambda,\mu}=1$, and $a_{\lambda,\mu}=0$ for all other weights λ. Some related properties on the multiplicities $a_{\lambda,\mu}$ are proved, and arguments in favour of our main conjecture are given. Finally, an extension of the conjecture describing the inverse of the matrix of Kazhdan-Lusztig polynomials is discussed.

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1 Introduction

Shortly after the classification of finite-dimensional simple Lie superalgebras [5, 11], Kac considered the problem of classifying all finite-dimensional simple modules (i.e. irreducible representations) of the basic classical Lie superalgebras [6]. For a subclass of these simple modules, known as "typical" modules, Kac derived a character formula closely analogous to the Weyl character formula for simple modules of simple Lie algebras [6]. The problem of obtaining a character formula for the remaining "atypical" modules has been the subject of intensive investigation, both in the mathematics and physics literature. Several partial solutions to this problem were given, e.g. for covariant or contravariant tensor representations [3, 1], for so-called generic representations [9], for singly atypical representations [2, 15, 16], or for tame representations [7]. Only recently a solution for the characters of simple $\mathfrak{gl}(m/n)$ modules has been given by Serganova [13] (with partial results in [12]). Thus the characters of all simple modules of type I Lie superalgebras are now in principle known, since for $C(n)$ the problem was already solved in [16]. For the series of Lie superalgebras $\mathfrak{q}(n)$, the solution of the character problem has been announced in [10].

Let us briefly describe some aspects of Serganova's solution, and the main results of the present paper.

The Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m/n)$ has a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ consistent with the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, in particular $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. A finite-dimensional $\mathfrak{g}_0$ module $L_\lambda(\mathfrak{g}_0)$ with highest weight λ is turned into a $\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ module by trivial $\mathfrak{g}_{+1}$ action, and then the Kac module [5, 6] is the induced module

$$V_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_{+1})} L_\lambda(\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}).$$

This is a finite-dimensional module which is simple when λ is typical, and indecomposable when λ is atypical. The characters of Kac modules are easy to determine. Any simple finite-dimensional $\mathfrak{g}$ module $L_\lambda = L_\lambda(\mathfrak{g})$ is the quotient of the Kac module V_λ by its unique maximal submodule. Since V_λ is indecomposable for atypical λ one cannot decompose it into irreducibles, but on the other hand V_λ has a Jordan-Hölder composition series. Let $a_{\lambda,\mu} = [V_\lambda : L_\mu]$ be the multiplicity of L_μ in the composition series of V_λ. Then

$$\mathrm{ch}\, V_\lambda = \sum_\mu a_{\lambda,\mu}\, \mathrm{ch}\, L_\mu.$$

With the canonical partial order on the set of weights, the matrix $A = (a_{\lambda,\mu})$ is lower triangular (the diagonal consisting of 1's). On the other hand, the character of L_λ can be written as an infinite "alternating sum" of characters of Kac modules [13], thus

$$\mathrm{ch}\, L_\lambda = \sum_\mu b_{\lambda,\mu}\, \mathrm{ch}\, V_\mu.$$

The matrix $B = (b_{\lambda,\mu})$ is also lower triangular, and A and B are inverses to each other. In a recent paper of Serganova [13], it is shown that the coefficients $b_{\lambda,\mu}$ are equal to the value of Kazhdan-Lusztig polynomials $K_{\lambda,\mu}(q)$ for $q = -1$, and an algorithm is given for evaluating $b_{\lambda,\mu} = K_{\lambda,\mu}(-1)$ using induction on dimension and the embedding $\mathfrak{gl}(1) \oplus \mathfrak{gl}(m-1/n) \subset \mathfrak{gl}(m/n)$. Serganova's work offers a principal solution to an outstanding problem which has been open for almost 20 years. However, there remain some important open questions related to the character problem for $\mathfrak{gl}(m/n)$. First, the methods of [13] offer a direct solution to finding the multiplicities $a_{\lambda,\mu}$, but to find the actual character $\operatorname{ch} L_\lambda$ (i.e. the coefficients $b_{\lambda,\mu}$) the method is rather indirect in the sense that it depends on an formal inversion process (see also [13, Remarks 2.4, 2.5 and Example 2.6]). For example, it would be difficult to extract from [13, Theorem 2.3] what the lowest weight μ is in the decomposition of L_λ with respect to the even subalgebra, i.e.

$$L_\lambda(\mathfrak{g}) = \bigoplus_{\mu} c_{\lambda,\mu} L_\mu(\mathfrak{g}_0).$$

Secondly, the algorithm presented in [13, Section 2], though straighforward, is not easy to apply, and does not shed much light on the properties of the coefficients $a_{\lambda,\mu}$. Basically, the algorithm starts with a given λ, and allows one in various steps to determine the weights μ for which $a_{\lambda,\mu}$ is possibly not zero; in general, however, many cancellations take place at the end of the calculation and after this one finishes with the actual μ's with non-zero $a_{\lambda,\mu}$.

The goal of the present Letter is to announce some results on the multiplicities $a_{\lambda,\mu}$. In particular, we claim that $a_{\lambda,\mu}$ is either 0 or 1. Moreover, we explain how the structure of the matrix $A = (a_{\lambda,\mu})$, which seems extremely complicated when rows are considered, is in fact very simple when concentrating on columns. In particular, we give an easy algorithm for finding the non-zero $a_{\lambda,\mu}$. Our algorithm is opposite in the sense that for given μ all λ are determined for which $a_{\lambda,\mu} \neq 0$, and in such case $a_{\lambda,\mu} = 1$; in all other cases $a_{\lambda,\mu} = 0$. We should immediately add that at the moment we have a proof of this property only in some cases, but our proof is not valid for all cases. However, with all the data deduced by means of Serganova's algorithm we feel safe that it is always valid. Based on this observation, we also present a conjecture giving the inverse matrix $A_q = (a_{\lambda,\mu}(q))$ of the matrix of Kazhdan-Lusztig polynomials $K_q = (K_{\lambda,\mu}(q))$. A strong argument in favour of this conjecture is presented, and some properties of Kazhdan-Lusztig polynomials are derived.

The emphasis of this paper is on announcing these results, and on proving some related properties. Some of the proofs involve combinatorial arguments; here we have chosen to be as concise as possible, so that the introduction of many combinatorial quantities [4] related to $\mathfrak{gl}(m/n)$ can be avoided.

2 Notation and definitions

Let $\mathfrak{g} = \mathfrak{gl}(m/n)$, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra, and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ the consistent $\mathbb{Z}$-grading. Note that $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, and put $\mathfrak{g}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ and $\mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. The dual space $\mathfrak{h}^*$ has a natural basis $\{\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n\}$, and the roots of $\mathfrak{g}$ can be expressed in terms of this basis. Let Δ be the set of all roots, Δ_0 the set of even roots, and Δ_1 the set of odd roots. One can choose a set of simple roots (or, equivalently, a triangular decomposition), but note that contrary to the case of simple Lie algebras not all such choices are equivalent. The distinguished choice for a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is such that $\mathfrak{g}_{+1} \subset \mathfrak{n}^+$ and $\mathfrak{g}_{-1} \subset \mathfrak{n}^-$. Then $\mathfrak{h} \oplus \mathfrak{n}^+$ is the corresponding distinguished Borel subalgebra, and Δ_+ the set of positive roots. For this choice we have explicitly :

$$\begin{aligned}
\Delta_{0,+} &= \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq m\} \cup \{\delta_i - \delta_j | 1 \leq i < j \leq n\}, \\
\Delta_{1,+} &= \{\beta_{ij} = \epsilon_i - \delta_j | 1 \leq i \leq m,\ 1 \leq j \leq n\},
\end{aligned}$$

and the corresponding set of simple roots is given by

$$\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}.$$

Thus in the distinguished basis there is only one simple root which is odd. As usual, we put

$$\rho_0 = \frac{1}{2}(\sum_{\alpha \in \Delta_{0,+}} \alpha), \qquad \rho_1 = \frac{1}{2}(\sum_{\alpha \in \Delta_{1,+}} \alpha), \qquad \rho = \rho_0 - \rho_1.$$

There is a symmetric form $(\ ,\ )$ on $\mathfrak{h}^*$ induced by the invariant symmetric form on $\mathfrak{g}$, and in the natural basis it takes the form $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$ and $(\delta_i, \delta_j) = -\delta_{ij}$. A weight $\lambda \in \mathfrak{h}^*$ is integral if $(\lambda, \alpha) \in \mathbb{Z}$ for all even roots α, and it is dominant if $2(\lambda, \alpha)/(\alpha, \alpha)$ is a nonnegative integer for all positive even roots α. Following Sergavona [13] we say that λ is regular if it is not on a wall in any Weyl chamber, i.e. if $(\lambda, \alpha) \neq 0$ for every α in Δ_0. The set of integral weights is denoted by P, the set of integral dominant weights by P^+. With our choice of positive roots, the weights P are partially ordered by $\lambda \leq \mu$ iff $\mu - \lambda = \sum k_\alpha \alpha$ where $\alpha \in \Delta_+$ and k_α nonnegative integers. In the standard ϵ-δ-basis, an integral weight $\lambda \in P$ is written as

$$\begin{aligned}
\lambda &= \lambda_1 \epsilon_1 + \cdots + \lambda_m \epsilon_m + \lambda'_1 \delta_1 + \cdots + \lambda'_n \delta_n, \\
&= (\lambda_1, \cdots, \lambda_m; \lambda'_1, \cdots, \lambda'_n),
\end{aligned}$$

where $\lambda_i - \lambda_{i+1} \in \mathbb{Z}$ and $\lambda'_i - \lambda'_{i+1} \in \mathbb{Z}$; $\lambda \in P^+$ if moreover $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_m$ and $\lambda'_1 \geq \lambda'_2 \geq \cdots \lambda'_n$.

The Weyl group of $\mathfrak{g}$ is the Weyl group W of $\mathfrak{g}_0$, hence it is the direct product of symmetric groups $S_m \times S_n$. For $w \in W$, we denote by $\varepsilon(w)$ its signature. The dot

action is defined as $w \cdot \lambda = w(\lambda+\rho)-\rho$, for $w \in W$ and $\lambda \in \mathfrak{h}^*$. For $\lambda \in P$, $W\lambda$ has a unique element in P^+ (i.e. in the dominant Weyl chamber), and this is denoted by $d(\lambda)$. If $d(\lambda+\rho)-\rho \in P^+$, then we define $\dot{d}(\lambda) = d(\lambda+\rho)-\rho$; if $d(\lambda+\rho)-\rho \notin P^+$, then $\dot{d}(\lambda)$ is said to be undefined.

For λ integral dominant, let $L_\lambda(\mathfrak{g}_0)$ denote the finite-dimensional irreducible $\mathfrak{g}_0$ module with highest weight λ. This can be extended to a $\mathfrak{g}^+$ module $L_\lambda(\mathfrak{g}^+)$ by letting $\mathfrak{g}_{+1}$ act trivially on the elements of $L_\lambda(\mathfrak{g}_0)$. The Kac module V_λ is then the finite-dimensional module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}^+)} L_\lambda(\mathfrak{g}^+)$. This is in general indecomposable, and the irreducible $\mathfrak{g}$ module obtained by quotienting V_λ by its maximal submodule is denoted by L_λ or $L_\lambda(\mathfrak{g})$. All these modules V are $\mathfrak{h}$-diagonalizable with weight decomposition $V = \oplus_\mu V(\mu)$, and the character is defined to be $\operatorname{ch} V = \sum_\mu \dim V(\mu)\, e^\mu$, where e^μ is the formal exponential.

Let $\lambda \in P^+$, then λ (resp. V_λ and L_λ) is said to be typical if $(\lambda+\rho,\alpha) \neq 0$ for every $\alpha \in \Delta_{1,+}$. Otherwise λ (resp. V_λ and L_λ) is said to be atypical. The number $r = \#\lambda$ of elements $\alpha \in \Delta_{1,+}$ for which $(\lambda+\rho,\alpha)=0$ is called the degree of atypicality. This definition depends upon the choice of Δ_+, but one can easily extend it such that the degree of atypicality of a simple $\mathfrak{g}$ module V does not depend upon this choice (see [7, Corollary 3.1]). If λ is typical, $L_\lambda = V_\lambda$, and the character is easy to write down :

$$\operatorname{ch} V_\lambda = \prod_{\alpha\in\Delta_{1,+}} (1+e^{-\alpha})\ \operatorname{ch} L_\lambda(\mathfrak{g}_0).$$

If λ is atypical then V_λ is indecomposable, and we denote by $a_{\lambda,\mu} = [V_\lambda : L_\mu]$ the multiplicity of L_μ in the composition series of V_λ, thus $\operatorname{ch} V_\lambda = \sum_\mu a_{\lambda,\mu} \operatorname{ch} L_\mu$. The multiplicities $a_{\lambda,\mu}$ can be nonzero only when $\mu \leq \lambda$ with respect to the partial order. Moreover $a_{\lambda,\lambda} = 1$ and if $\#\mu \neq \#\lambda$ then $a_{\lambda,\mu} = 0$, see [13].

3 Preliminaries

In this section (and the following one) μ is an integral dominant r-fold atypical weight ($\#\mu = r$), and the r elements α of $\Delta_{1,+}$ for which $(\mu+\rho,\alpha)=0$ are denoted by $\{\gamma_1,\dots,\gamma_r\}$. Moreover, they are ordered in such a way that

$$\gamma_1 < \gamma_2 < \cdots < \gamma_r.$$

Lemma 3.1 *Let μ be integral dominant and $\alpha \in \Delta_{1,+}$ such that $(\mu+\rho,\alpha)=0$. Then there exists a unique maximal subset $\{\alpha=\alpha_0,\alpha_1,\dots,\alpha_k\}$ of $\Delta_{1,+}$ with every $\alpha_i > \alpha$ (for $i>0$) such that*

$$(\mu+\rho,\alpha_0)=0, (\mu+\alpha_0+\rho,\alpha_1)=0,\dots,(\mu+\alpha_0+\cdots+\alpha_{k-1}+\rho,\alpha_k)=0.$$

Moreover, $\lambda = \mu + \alpha_0 + \cdots + \alpha_{k-1} + \alpha_k$ *is integral dominant.*

The set is maximal in the sense that any element $\alpha_{k+1} > \alpha$ for which $(\lambda+\rho, \alpha_{k+1}) = 0$ belongs already to the set.

Proof. The proof is combinatorial, and we do not give it here. It follows exactly the same arguments as in [8], or as in [4, Definition 6.1 and Theorem 6.2]. $\square$

Definition 3.2 *Let* μ *be integral dominant and* r*-fold atypical with respect to the roots* $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ $(\gamma_i \in \Delta_{1,+})$. *For each* γ_i*, let* $\Delta(\gamma_i)$ *denote the subset of* $\Delta_{1,+}$ *defined by Lemma 3.1.*

It is easy to deduce that

$$\Delta(\gamma_1) \supset \Delta(\gamma_2) \supset \cdots \supset \Delta(\gamma_r).$$

Example 3.3 Let $\mathfrak{g} = \mathfrak{gl}(4/5)$, and $\mu = (2,1,0,0;0,-2,-2,-2,-2)$. The numbers $(\mu+\rho, \beta_{i,j})$ $(\beta_{i,j} = \epsilon_i - \delta_j)$ are given here in the atypicality matrix [14, 15] $A(\mu)$:

$$A(\mu) = \begin{pmatrix} 5 & 2 & 1 & 0 & -1 \\ 3 & 0 & -1 & -2 & -3 \\ 1 & -2 & -3 & -4 & -5 \\ 0 & -3 & -4 & -5 & -6 \end{pmatrix}.$$

Thus $\#\mu = 3$, $\gamma_1 = \beta_{4,1}$, $\gamma_2 = \beta_{2,2}$, and $\gamma_3 = \beta_{1,4}$. One can verify that :

$$\begin{aligned} \Delta(\gamma_3) &= \{\beta_{1,4}, \beta_{1,5}\}, \\ \Delta(\gamma_2) &= \{\beta_{2,2}, \beta_{2,3}, \beta_{2,4}, \beta_{2,5}, \beta_{1,3}, \beta_{1,4}, \beta_{1,5}\}, \\ \Delta(\gamma_1) &= \{\beta_{4,1}, \beta_{3,1}, \beta_{2,2}, \beta_{2,3}, \beta_{2,4}, \beta_{2,5}, \beta_{1,3}, \beta_{1,4}, \beta_{1,5}\}. \end{aligned}$$

Since $\Delta(\gamma_i) \supset \Delta(\gamma_{i+1})$ we can consider their differences.

Definition 3.4 *Let* μ *be integral dominant and* r*-fold atypical with respect to the roots* $\gamma_1 < \gamma_2 < \cdots < \gamma_r$*, and* $\Delta(\gamma_i)$ *as in Definition 3.2. Then* $\nabla(\gamma_i) = \Delta(\gamma_i)\backslash\Delta(\gamma_{i+1})$ *(with* $\nabla(\gamma_r) = \Delta(\gamma_r)$*). Denote by* $k_i = \#\nabla(\gamma_i)$ *the number of elements in* $\nabla(\gamma_i)$*. Two such roots* γ_i *and* γ_j *are said to be disconnected for* μ *if* $\nabla(\gamma_i) \perp \nabla(\gamma_j)$ *(orthogonal with respect to the symmetric form* $(\ ,\ )$ *on* $\mathfrak{h}^*$*); otherwise they are connected for* μ*.*

In example 3.3, we have

$$\begin{aligned} \nabla(\gamma_3) &= \{\beta_{1,4}, \beta_{1,5}\}, \\ \nabla(\gamma_2) &= \{\beta_{2,2}, \beta_{2,3}, \beta_{2,4}, \beta_{2,5}, \beta_{1,3}\}, \\ \nabla(\gamma_1) &= \{\beta_{4,1}, \beta_{3,1}\}, \end{aligned}$$

hence $(k_1, k_2, k_3) = (2,5,2)$; γ_1 and γ_2 (or γ_3) are disconnected, whereas γ_2 and γ_3 are connected for μ.

Proposition 3.5 *Let μ be integral dominant and r-fold atypical with respect to the roots $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ ($\gamma_i \in \Delta_{1,+}$). Consider the decomposition of L_μ with respect to the even subalgebra :*

$$L_\mu = L_\mu(\mathfrak{g}) = \bigoplus_\nu c_{\mu,\nu} L_\nu(\mathfrak{g}_0).$$

The set $S = \{\nu \in P^+ | c_{\mu,\nu} \neq 0\}$ contains a unique smallest element μ_0 (with respect to the partial order), $c_{\mu,\mu_0} = 1$, and

$$\mu_0 = \mu - \sum_{\alpha \in \Delta_{1,+} \setminus \Delta(\gamma_1)} \alpha.$$

Proof. We need the notion of reflection with respect to an odd simple root [8, 7]. For a given subset of positive roots Δ_+ of Δ and an odd simple root α, one may construct a new subset of positive roots Δ'_+ by

$$\Delta'_+ = (\Delta_+ \cup \{-\alpha\}) \setminus \{\alpha\}.$$

The set Δ'_+ is said to be obtained from Δ_+ by a simple α-reflection. Note that the set of even positive roots $\Delta_{0,+}$ remains unchanged. In that case, the new ρ is given by

$$\rho' = \rho + \alpha.$$

When V is a finite-dimensional simple $\mathfrak{g}$ module, and Δ_+ is a fixed set of positive roots, then V has a highest weight λ with respect to Δ_+. If Δ'_+ is obtained from Δ_+ by a simple α-reflection (α odd), then the highest weight λ' of V with respect to Δ'_+ is given by [7, (3.1)] :

$$\begin{array}{lll} \lambda' = \lambda - \alpha & \text{if} & (\lambda|\alpha) \neq 0 \\ \lambda' = \lambda & \text{if} & (\lambda|\alpha) = 0. \end{array}$$

Moreover, one has

$$\begin{array}{lll} \lambda' + \rho' = \lambda + \rho & \text{if} & (\lambda + \rho|\alpha) \neq 0 \\ \lambda' + \rho' = \lambda + \rho + \alpha & \text{if} & (\lambda + \rho|\alpha) = 0. \end{array}$$

Now we apply this to $V = L_\mu(\mathfrak{g})$, starting with the distinguished positive set of roots Δ_+ given in Section 2, thus the highest weight with respect to Δ_+ is μ. The only odd simple root is $\alpha = \beta_{m,1} = \epsilon_m - \delta_1$; applying the simple α-reflection leads to a new set of positive roots Δ'_+ and the new highest weight μ' is given by $\mu' = \mu - \alpha$ if $(\mu + \rho|\alpha) \neq 0$ or by $\mu' = \mu$ if $(\mu + \rho|\alpha) = 0$ (for the current α we have that $(\rho|\alpha) = 0$). Next, apply the simple α'-reflection with respect to $\alpha' = \beta_{m,2} = \epsilon_m - \delta_2$ (which is indeed an odd simple root in Δ'_+) to obtain the new set Δ''_+. We have $(\mu'|\alpha') = (\mu' + \rho'|\alpha')$. So

- if $(\mu+\rho|\alpha) \neq 0$ then
  - if $(\mu+\rho|\alpha') = (\mu'+\rho'|\alpha') = (\mu'|\alpha') \neq 0$ then $\mu'' = \mu' - \alpha' = \mu - \alpha - \alpha'$,
  - if $(\mu+\rho|\alpha') = 0$ then $\mu'' = \mu' = \mu - \alpha$;
- if $(\mu+\rho|\alpha) = 0$ then
  - if $(\mu+\alpha+\rho|\alpha') = (\mu'+\rho'|\alpha') = (\mu'|\alpha') \neq 0$ then $\mu'' = \mu' - \alpha' = \mu - \alpha'$,
  - if $(\mu+\alpha+\rho|\alpha') = 0$ then $\mu'' = \mu' = \mu$.

So far we have applied only two simple α-reflections, namely first with respect to $\beta_{m,1}$ and then with respect to $\beta_{m,2}$. Now continue applying such reflections, with respect to (in this order)

$$\beta_{m,3}, \ldots, \beta_{m,n}, \beta_{m-1,1}, \beta_{m-1,2}, \ldots, \beta_{m-1,n}, \ldots, \ldots, \beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,n}.$$

Using lemma 3.1 and definition 3.2, it is an easy combinatorial exercise to see that with respect to the final set of positive roots, the highest weight of our module V is given by $\mu_0 = \mu - \sum_{\alpha\in\Delta_{1,+}\setminus\Delta(\gamma_1)} \alpha$, since in this process an odd root α will be subtracted only when it is not in $\Delta(\gamma_1)$. The final set of positive roots is given by

$$\Delta_{f,+} = \Delta_{0,+} \cup \{-\beta_{i,j} = \delta_j - \epsilon_i | 1 \leq i \leq m,\ 1 \leq j \leq n\},$$

where $\Delta_{0,+}$ remains unchanged. But at every stage in the process, the new highest weight ν is unique and satisfies $c_{\mu,\nu} = 1$. The final weight μ_0 is the highest weight of V with respect to $\Delta_{f,+}$, so it must be the smallest element in S. □

In example 3.3 we have $\mu_0 = (0,0,-4,-4;2,1,0,0,0)$. This is the lowest $\mathfrak{g}_0$ highest weight in the decomposition of L_μ with respect to the even subalgebra $\mathfrak{g}_0$.

Proposition 3.6 *Let μ be integral dominant and r-fold atypical with respect to the roots $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ ($\gamma_i \in \Delta_{1,+}$), and*

$$\mu_0 = \mu - \sum_{\alpha\in\Delta_{1,+}\setminus\Delta(\gamma_1)} \alpha.$$

Let $k_i = \#\nabla(\gamma_i)$ (see Definition 3.4). Then

$$\dot{d}(\mu + \sum_{i=1}^{r} k_i\gamma_i) = \mu_0 + 2\rho_1.$$

Proof. The proof is combinatorial and uses a number of notions defined in [4]. With the weight μ there corresponds a composite Young diagram, specified by the Young diagrams of the two parts of μ associated to $\mathfrak{gl}(m)$ and $\mathfrak{gl}(n)$ respectively. The

addition of a coordinated boundary strip of length k to μ, starting at the position of γ_i, was discussed in [4], and the corresponding weight is given by $\dot{d}(\mu + k_i\gamma_i)$ if the resulting composite diagram is standard (if the resulting composite diagram is not standard, $\dot{d}$ is undefined on $\mu + k_i\gamma_i$). Using the arguments of [4, Theorem 6.2] or of [15, Lemma 6.7] one can deduce that

$$\dot{d}(\mu + k_r\gamma_r) = \mu + \sum_{\alpha\in\nabla(\gamma_r)} \alpha \qquad \in P^+,$$

and then

$$\begin{aligned} \dot{d}(\mu + \sum_{i=1}^{r} k_i\gamma_i) &= \mu + \sum_{\alpha\in\nabla(\gamma_1)} \alpha + \sum_{\alpha\in\nabla(\gamma_2)} \alpha + \cdots + \sum_{\alpha\in\nabla(\gamma_r)} \alpha \\ &= \mu + \sum_{\alpha\in\Delta(\gamma_1)} \alpha. \end{aligned}$$

Using μ_0 and the definition of ρ_1, the final result follows. □

In example 3.3,

$$\mu + \sum_{i=1}^{r} k_i\gamma_i = \mu + 2\beta_{4,1} + 5\beta_{2,2} + 2\beta_{1,4} = (4,6,0,2;-2,-7,-2,-4,-2).$$

Applying $\dot{d}$ to this weight gives $(5,5,1,1;-2,-3,-4,-4,-4)$, which is indeed $\mu_0 + 2\rho_1$.

4 Main results

Suppose again that μ is integral dominant and r-fold atypical with respect to the roots $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ ($\gamma_i \in \Delta_{1,+}$). The Kac module $V_{\mu_0+2\rho_1}$, with highest weight $\mu_0 + 2\rho_1$, is completely reducible with respect to $\mathfrak{g}_0$. From the structure of Kac modules it follows that in this reduction $V_{\mu_0+2\rho_1}$ has a unique $\mathfrak{g}_0$ component with highest weight μ_0. In fact, the highest weight vector v_{μ_0} of this component is obtained by applying the product of all negative odd root vectors to the highest weight vector $v_{\mu+2\rho_1}$ of the Kac module [6, 15]. This vector v_{μ_0} is contained in every submodule of $V_{\mu_0+2\rho_1}$. Then it follows from proposition 3.5 that every $\mathfrak{g}$ submodule of $V_{\mu_0+2\rho_1}$ must also contain L_μ, in other words L_μ is the smallest $\mathfrak{g}$ submodule in the Kac module $V_{\mu_0+2\rho_1}$.

Thus, starting from μ one determines the weight $\lambda = \dot{d}(\mu + \sum_{i=1}^{r} k_i\gamma_i)$. Then V_λ has L_μ as a composition factor, $a_{\lambda,\mu} = 1$, and L_μ is the smallest $\mathfrak{g}$ submodule in V_λ. Moreover, for every $\nu > \lambda$, we have that $a_{\nu,\mu} = 0$. Indeed, V_ν cannot have L_μ

as a composition factor since the smallest $\mathfrak{g}_0$ highest weight in V_ν is $\nu - 2\rho_1$, and $\nu - 2\rho_1 > \mu_0$.

Hence, all weights λ for which $a_{\lambda,\mu} \neq 0$ must lie between $\mu \leq \lambda \leq \dot{d}(\mu + \sum_{i=1}^r k_i\gamma_i)$. The following is our main result : we give a simple expression for those λ with $a_{\lambda,\mu} \neq 0$:

Conjecture 4.1 *Let μ be integral dominant and r-fold atypical with respect to the roots $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ ($\gamma_i \in \Delta_{1,+}$), and $k_i = \#\nabla(\gamma_i)$ (see definition 3.4). For $\theta = (\theta_1, \ldots, \theta_r) \in \{0,1\}^r$, consider*

$$\lambda_\theta = \dot{d}(\mu_\theta) = \dot{d}(\mu + \sum_{i=1}^r \theta_i k_i \gamma_i).$$

Then $a_{\lambda_\theta,\mu} = 1$ for each of the 2^r integral dominant weights λ_θ, and $a_{\lambda,\mu} = 0$ elsewhere.

There are a number of cases in which the conjecture can be proved. For $r = 0$ it is trivial; for $r = 1$ it follows from the results of [15]. For generic weights μ it can be deduced from [8, 9]. When all roots γ_i are disconnected for μ, the conjecture can be proved by induction on r. But for the general case we do not have a proof so far. For every given λ, $a_{\lambda,\mu}$ can be calculated from Serganova's algorithm [13], and thus we were able to successfully verify this conjecture for numerous examples. In Serganova's approach, induction from $\mathfrak{gl}(1) \oplus \mathfrak{gl}(m-1/n)$ to $\mathfrak{gl}(m/n)$ is used, and in this setting the question of finding those μ with $a_{\lambda,\mu} \neq 0$ for given λ is natural; but the question of finding those λ with $a_{\lambda,\mu} \neq 0$ for given μ is rather unnatural and we think it cannot be solved directly using the same type of induction.

Finally, note that the above conjecture is closely related to [4, Conjecture 7.2].

Remark 4.2 Since BGG duality holds for $\mathfrak{gl}(m/n)$ [17, Theorem 2.7], it follows that $a_{\lambda,\mu}$ also describes the multiplicity of the Kac module V_λ in a Kac composition series of the indecomposable projective module I_μ [17, §2]. Thus according to Conjecture 4.1 the multiplicities of Kac modules in indecomposable projective modules are easier to describe than the multiplicities of simple modules in Kac modules.

Example 4.3 Take the data from example 3.3, i.e. $\mathfrak{g} = \mathfrak{gl}(4/5)$, and $\mu = (2,1,0,0;0,-2,-2,-2,-2)$. We know already that $r = 3$, $(\gamma_1, \gamma_2, \gamma_3) =$

$(\beta_{4,1}, \beta_{2,2}, \beta_{1,4})$, and $(k_1, k_2, k_3) = (2, 5, 2)$. It is easy to calculate the μ_θ and λ_θ :

$$\begin{array}{lll}
\theta\ : & \mu_\theta\ : & \lambda_\theta\ : \\
(0,0,0) & (2,1,0,0;0,-2,-2,-2,-2) & (2,1,0,0;0,-2,-2,-2,-2) \\
(1,0,0) & (2,1,0,2;-2,-2,-2,-2,-2) & (2,1,1,1;-2,-2,-2,-2,-2) \\
(0,1,0) & (2,6,0,0;0,-7,-2,-2,-2) & (5,3,0,0;0,-3,-3,-3,-4) \\
(0,0,1) & (4,1,0,0;0,-2,-2,-4,-2) & (4,1,0,0;0,-2,-2,-3,-3) \\
(1,1,0) & (2,6,0,2;-2,-7,-2,-2,-2) & (5,3,1,1;-2,-3,-3,-3,-4) \\
(1,0,1) & (4,1,0,2;-2,-2,-2,-4,-2) & (4,1,1,1;-2,-2,-2,-3,-3) \\
(0,1,1) & (4,6,0,0;0,-7,-2,-4,-2) & (5,5,0,0;0,-3,-4,-4,-4) \\
(1,1,1) & (4,6,0,2;-2,-7,-2,-4,-2) & (5,5,1,1;-2,-3,-4,-4,-4)
\end{array}$$

So all Kac modules V_{λ_θ} have L_μ as a composition factor, with multiplicity 1, and $a_{\lambda,\mu} = 0$ for all other λ. The first weight is $\lambda_{(0,0,0)} = \mu$, and obviously $a_{\mu,\mu} = 1$; the last weight is $\lambda_{(1,1,1)} = \mu_0 + 2\rho_1$, and here we have proved earlier that indeed $a_{\mu_0+2\rho_1,\mu} = 1$.

We shall now consider a number of consequences of Conjecture 4.1. First, the matrix $A = (a_{\lambda,\mu})$ is now easy to determine, column by column. With $\#\mu$ denoting the degree of atypicality, we have

$$\sum_\lambda a_{\lambda,\mu} = 2^{\#\mu}$$

for every $\mu \in P^+$. Recall that the inverse matrix $B = (b_{\lambda,\mu})$ of the lower triangular matrix A consists of the coefficients in the character formula for L_λ :

$$\mathrm{ch}\, L_\lambda = \sum_\mu b_{\lambda,\mu}\, \mathrm{ch}\, V_\mu.$$

These coefficients are equal to specializations of Kazhdan-Lusztig polynomials $K_{\lambda,\mu}(q)$, i.e. $b_{\lambda,\mu} = K_{\lambda,\mu}(-1)$. Kazhdan-Lusztig polynomials for $\mathfrak{gl}(m/n)$ were defined by Serganova [12, 13]. Consider the ith homology $H_i(\mathfrak{g}_{-1}; L_\lambda)$. This space has the structure of a $\mathfrak{g}_0$ module, and denote the multiplicity $[H_i(\mathfrak{g}_{-1}; L_\lambda) : L_\mu(\mathfrak{g}_0)]$ by $K^i_{\lambda,\mu}$. Then the Kazhdan-Lusztig polynomials are defined as

$$K_{\lambda,\mu}(q) = \sum_{i=0}^{\infty} K^i_{\lambda,\mu} q^i.$$

Since $H_i(\mathfrak{g}_{-1}; L_\lambda)$ is a $\mathfrak{g}_0$ quotient module of $\mathrm{Sym}^i(\mathfrak{g}_{-1}) \otimes L_\lambda$, the $K_{\lambda,\mu}(q)$'s are polynomials (and not infinite series) in q.

In general $H_i(\mathfrak{g}_{-1}; L_\lambda)$ is difficult to determine, except when $\lambda = 0$ because in that case

$$H_i(\mathfrak{g}_{-1}; L_0) = \mathrm{Sym}^i(\mathfrak{g}_{-1}).$$

One can explicitly construct the decomposition with respect to $\mathfrak{g}_0$ for these modules (assume $m \leq n$) :

$$\mathrm{Sym}^i(\mathfrak{g}_{-1}) = \bigoplus_{\sigma} L_{(-\sigma_m,\ldots,-\sigma_2,-\sigma_1;\sigma_1,\sigma_2,\ldots,\sigma_m,0,\ldots,0)}(\mathfrak{g}_0),$$

where the sum is over those partitions σ with $|\sigma| = i$ and with m parts, i.e. over all integers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$ with $\sigma_1 + \sigma_2 + \cdots + \sigma_m = i$. Thus

$$K_{0,(-\sigma_m,\ldots,-\sigma_2,-\sigma_1;\sigma_1,\sigma_2,\ldots,\sigma_m,0,\ldots,0)}(q) = q^{\sigma_1+\sigma_2+\cdots+\sigma_m}$$

if σ is a partition with m parts, and $K_{0,\mu}(q) = 0$ for all other weights μ. Observe that this result is in agreement with the expansion (8.8) of [14] for $q = -1$.

Since we have a simple way to calculate the matrix $A = (a_{\lambda,\mu})$, being the inverse of $B = (b_{\lambda,\mu}) = (K_{\lambda,\mu}(-1))$, it would be interesting to see if also the inverse of $K_q = (K_{\lambda,\mu}(q))$, say $A_q = (a_{\lambda,\mu}(q))$, can easily be determined. Here, we have the following :

Conjecture 4.4 *Let μ be integral dominant and r-fold atypical with respect to the roots $\gamma_1 < \gamma_2 < \cdots < \gamma_r$ ($\gamma_i \in \Delta_{1,+}$). Consider the 2^r integral dominant weights λ_θ, $\theta = (\theta_1, \ldots, \theta_r) \in \{0,1\}^r$, determined in conjecture 4.1. Let*

$$a_{\lambda_\theta,\mu}(q) = (-q)^{|\theta|}, \qquad |\theta| = \sum_i \theta_i,$$

and $a_{\lambda,\mu}(q) = 0$ for all other λ. Then the inverse of the triangular matrix $A_q = (a_{\lambda,\mu}(q))$ is the matrix of Kazhdan-Lusztig polynomials $K_q = (K_{\lambda,\mu}(q))$.

We have no general proof, but only a number of consistency checks. For this purpose, it is useful to consider in A_q and K_q the submatrices corresponding to r-fold atypical weights (there is no overlap in these submatrices, i.e. if $\#\lambda \neq \#\mu$ then $a_{\lambda,\mu}(q) = 0 = K_{\lambda,\mu}(q)$.) For the submatrices corresponding to typical weights, the conjecture obviously holds, since both these submatrices are the identity matrix. For the submatrices corresponding to singly atypical weights ($r = 1$), it follows from [15] and [17] that the conjecture is true. As an additional verification, we have considered the determination of $K_{0,\mu}(q)$ by explicitly inverting the matrix $(a_{\lambda,\mu}(q))$ for some $\mathfrak{gl}(m/n)$ with small values of m and n :

Example 4.5 Let $\mathfrak{g} = \mathfrak{gl}(2/2)$, and consider the integral dominant weights μ with $\#\mu = 2$. These are of the form $(x, y; -y, -x)$, x and y integers with $x \geq y$. The nonzero multiplicities are given by

- if $x = y$ then $a_{(x,y;-y,-x),(x,y;-y,-x)} = 1$, $a_{(x,y;-y,-x),(x,y-1;-y+1,-x)} = -q$, and $a_{(x,y;-y,-x),(x-2,y-2;-y+2,-x+2)} = q^2$.

- if $x = y+1$ then $a_{(x,y;-y,-x),(x,y;-y,-x)} = 1$, $a_{(x,y;-y,-x),(x,y-1;-y+1,-x)} = -q$, $a_{(x,y;-y,-x),(x-1,y;-y,-x+1)} = -q$, $a_{(x,y;-y,-x),(x-2,y-1;-y+1,-x+2)} = -q$, and $a_{(x,y;-y,-x),(x-1,y-1;-y+1,-x+1)} = q^2$.
- if $x \geq y+2$ then $a_{(x,y;-y,-x),(x,y;-y,-x)} = 1$, $a_{(x,y;-y,-x),(x-1,y;-y,-x+1)} = -q$, $a_{(x,y;-y,-x),(x,y-1;-y+1,-x)} = -q$, and $a_{(x,y;-y,-x),(x-1,y-1;-y+1,-x+1)} = q^2$.

Note that these are also the values one would find by applying Serganova's algorithm [13] without the specialization $q=-1$. Using these values for $a_{\lambda,\mu}(q)$, one can calculate explicitly the matrix elements of the inverse of A_q. In particular, we have determined the values of $(A_q^{-1})_{0,\mu}$, and found

$$(A_q^{-1})_{0,(-x,-y;y,x)} = q^{x+y}, \qquad y \geq x \geq 0 \qquad (x, y \in \mathbb{Z}),$$

and 0 elsewhere. This coincides with the known values of $K_{0,\mu}(q)$. We have constructed only one row of the inverse matrix (namely where we can compare the answer), but this on its own is already a rather strong argument in favour of the conjecture since determining this row involves the knowledge of "all" elements of A_q.

Just as conjecture 4.1 has some interesting consequences, also the present conjecture has nice implications, in particular :

$$\sum_\lambda a_{\lambda,\mu}(q) = (1-q)^{\#\mu}$$

for every $\mu \in P^+$. And therefore also :

$$\sum_\lambda K_{\lambda,\mu}(q) = \frac{1}{(1-q)^{\#\mu}}.$$

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